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1994 J. Phys. A: Math. Gen. 27 2335

(http://iopscience.iop.org/0305-4470/27/7/015)

J. Phys. A: Math. Gen. 27 (1994) 2335–2349. Printed in the UK

# A system of coupled oscillators can have arbitrary prescribed attractors

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Received 7 June 1993, in final form 16 November 1993

**Abstract.** A system of coupled nonlinear oscillators is considered. If the size $N$ of this oscillator network is large enough, then the model can have arbitrary prescribed attractors which can be defined by some finite-dimensional flows. One can effectively find a simple interaction between the oscillators which gives these prescribed attractors.

## 1. Introduction

The investigation of attractors for nonlinear dissipative systems is an important but difficult problem. Excepting for so-called monotone systems [1–4] and systems possessing Lyapunov functions [5, 6], this problem is not solved analytically and usually here one applies computer calculations.

For example, a lot of recent works [7–11] describe networks of associative memory. One can consider such networks as nonlinear dissipative systems which have a lot of local attractors. In the simplest cases these attractors are steady states (stable rest points in a phase space). Certainly, there are non-trivial situations which are more complicated. The standard approach is by use of computer calculations that allow us to investigate non-trivial attractors.

In this paper another approach is suggested: we solve an inverse problem. For some prescribed attractors, one can find classes of dissipative systems which have these attractors. Namely, let $\mathscr{U}_1 \ldots \mathscr{U}_p$ be prescribed attractors defined by ordinary differential equations (ODE)

$$\mathrm{d}y/\mathrm{d}t = Y^{(k)}(y) \qquad y = (y_1 \ldots y_n) \tag{1.1}$$

where $Y^{(k)}$ are fields on $\mathbb{R}^n$. The main result asserts that one can construct some special classes of nonlinear dissipative systems (similar to neural networks or coupled oscillator systems). These systems have the prescribed attractors $\mathscr{U}_i$ if model parameters are chosen in a special way. More exactly, one has attractors $\tilde{\mathscr{U}}_i$ close to prescribed ones. The sets $\tilde{\mathscr{U}}$ are connected with $\mathscr{U}$ by homeomorphisms which are close to the identity. Certainly, if $n > 2$, then the prescribed attractors can be chaotic or periodic.

If the size of our system $N$ (the number of oscillators) tends to infinity then the number $P$ (which defines the number of prescribed local attractors) also tends to infinity although generally $P \ll N$.

0305-4470/94/072335+15\$19.50  

The paper is organized as follows. In section 2 we formulate the model, prove the existence of solutions and discuss connections with previous classical studies. In section 3 we investigate the local attractors of the unperturbed model and describe the standard perturbation theory. In section 4 we solve the inverse problem. We find the coefficients $J$ giving the prescribed attractors. Finally, section 5 contains a discussion.

To conclude this introduction, we point out that there exists a very simple general method which allows one to construct different examples of system with a prescribed large time behaviour (with prescribed local attractors). Let us take the evolution equation in a Banach (or Hilbert) space $H$

$$u_t = \nu A u + F(u) + \lambda g(u) \tag{1.2}$$

under some natural assumption on linear operator $A$ and nonlinearities $F$ and $G$. Suppose unperturbed equation (1.2) (with $\lambda=0$) has some *a priori* bounded Lyapunov function [5] and also is invariant under some symmetry group $\Gamma$. As for $g$, here, on the contrary, one assumes that $g$ is non-invariant. Then one can expect that the unperturbed equation has some invariant manifolds consisting of equilibria. If the perturbation $g$ acts, then at these manifolds $M$ a complicated motion can occur. Choosing the appropriate $g$ one can create different complicated attractors or invariant sets [12–14]. For example, in [13, 14] new nonlinear effects are described for reaction–diffusion systems, in particular, localized chaotic nonlinear waves [13] and complicated bifurcations between different hyperbolic sets [14] (arising when $\nu$ varies).

Qualitatively, for small $\lambda$ the time behaviour of solutions (1.2) can be described very simply (although rigorous mathematical proof of such a picture can be difficult). During time interval $[0, T]$ (where $T \gg O(1)$ and $T \ll O(\lambda^{-1})$), the evolution $u(t)$ can be described by the unperturbed equation. As a result, this solution holds at some unperturbed stable equilibrium manifold $M$. For $t \gg O(\lambda^{-1})$, the perturbation $g$ should be taken into the consideration, thus, there arises a finite-dimensional flow (the dimension is equal to $\mathrm{Dim}\, M$).

To obtain non-trivial results, here it is useful to take a large symmetry group, for instance $[U(1)]^N$. Such a group occurs in the coupled oscillator system considered below.

## 2. Model. Phase space and existence

Our model has the form

$$\mathrm{d}u_j(t)/\mathrm{d}t = f(|u_j|^2)u_j(t) + \lambda_j \left\{ \sum_k [\mathrm{i}J^1_{jk}u_k(t) + J^2_{jk}q(u_k(t))] \right\} u_j(t) \qquad u_j(t) \in \mathbb{C} \tag{2.1}$$

where $u_j(t)$ are unknown complex functions, $u = (u_1(t), u_2(t) \ldots u_N(t))$, the index $j$ runs on some subdomain $\Omega$ of the lattice $\mathbb{Z}^n$ and where $|\Omega| = N$. For any node $j$ the values $\lambda_j$ are chosen so that the contribution of $\lambda$ in (2.1) is a small perturbation with respect to the term $f$. One assumes that

$$\Phi(z) = \int_0^z f(s)\, \mathrm{d}s \qquad f \in C^2 \tag{2.2}$$

and the potential $\Phi$ has a simple form with a single positive local maximum at $z=1$. For example, one can suppose that

$$f(0) < 0 \qquad f(1) = 0 \qquad f'(1) < 0 \qquad f''(z) \leqslant 0 \qquad \text{for any } z > 0. \tag{2.3}$$

This means, in particular, that $\Phi$ has an upper bound. The coefficients $J$ are real

$$J^1_{kj},\ J^2_{kj} \in \mathbb{R}. \tag{2.4}$$

Finally, one supposes that $q$ has a more exotic form. Namely, the function $q(\operatorname{Re} u, \operatorname{Im} u): \mathbb{R}^2 \to \mathbb{C}$ satisfies the following conditions

$$\text{(a)} \qquad |q(u)| < C|u| \qquad q \in C^2 \tag{2.5}$$

$$\text{(b)} \qquad q(u) \equiv \ln(u) \qquad \text{for } u \in V \tag{2.6}$$

where $V$ is some neighbourhood of circle $|u| = 1$. Notice that function $q$ is not an analytic function of complex variable $u$, however, it is a smooth function of $\operatorname{Re} u$, $\operatorname{Im} u$.

The coefficients of $J$ can be large (this will be important below). Thus, to restrict the perturbation, we should assume that

$$\lambda_j = \varepsilon_j \left(1 + \sum_k (|J^1_{jk}| + |J^2_{jk}|)\right)^{-1} \tag{2.7}$$

where

$$0 < \varepsilon_j < \varepsilon \qquad \text{for any } j \tag{2.8}$$

where $\varepsilon$ is a small parameter.

One considers (2.1), under these assumptions, in the metric phase space

$$U_C = \{\boldsymbol{u} : |u_j| < C \text{ for all } j\} \tag{2.9}$$

where $C$ is an arbitrary constant, $C > 1$. The topology of $U_C$ is defined by the norm

$$|\boldsymbol{u}| = \max |u_j|. \tag{2.10}$$

The following assertion shows that $U_c$ really is the appropriate phase space.

*Proposition 2.1.* Assume conditions (2.2–2.8) hold and $C > 1$. If initial data $\boldsymbol{v}$

$$v_j = u_j(0) \tag{2.11}$$

lie in $U_c$, and $\varepsilon$ is small enough, then the solution $\boldsymbol{u}(t)$ is *a priori* bounded

$$\boldsymbol{u}(t) \in U_C \qquad \text{for all } t > 0. \tag{2.12}$$

Thus, equations (2.1) define a global flow in $U_C$.

*Proof.* See appendix.

To conclude this section, let us discuss connections with some classical physical models. First one notices that one can add to the right-hand side of (2.1) the discrete Laplacian $d\Delta u$. For example, when $\Omega \subseteq \mathbb{R}$ one has $\Delta u_i = u_{i+1} - 2u_i + u_{i-1}$. Then again key proposition 2.1 holds. If $d$ is small enough, the solution behaviour will be analogous.

In this case the unperturbed equation reminds one of the discrete variants of the classical TDGL equation (time-dependent Ginzburg–Landau equation)

$$u_t = d\Delta u + f(|u|^2)u. \tag{2.13}$$

This fundamental model was investigated by classical works [15, 16]. A continuous variant of our model is an evolution equation with the non-local interaction

$$\partial u(x,t)/\partial t = d\Delta u(x,t) + f(u(x,t))u(x,t) + \int J(x,y), u(y))\,\mathrm{d}y. \tag{2.14}$$

Such equations (and even more complicated versions) can occur in polymer theory [17, 18]. The result obtained can probably be generalized in this case, however, this is a more difficult problem.

## 3. Attractor of unperturbed model. Perturbation theory

Due to assumptions (2.3), the unperturbed model ($\varepsilon=0$) can easily be investigated. As follows from (2.3) and assumptions on $\Phi$, the function $f$ has a single root $\xi$

$$f(\xi)=0 \qquad 0<\xi<1. \tag{3.1}$$

If $\varepsilon=0$ system (2.1) reduces to the independent equations

$$u_t = f(|u|^2)u. \tag{3.2}$$

It is easy to see that module $r=|u|$ satisfies

$$r_t = f(r^2)r. \tag{3.3}$$

Thus, equation (3.2) has two attractors $B_0=\{|u|=0\}$ and $B_1=\{|u|=1\}$ with the attraction basins, respectively

$$\mathscr{B}_1=\{0<|u|^2<\xi\} \qquad \mathscr{B}_2=\{|u|^2>\xi\}. \tag{3.4}$$

Returning to (2.1) with $\varepsilon=0$, one obtains that the unperturbed equation has a number of attractors $B(\Omega')$. Each attractor is defined by some subset $\Omega'$ of $\Omega$ and has the form

$$B(\Omega')=\begin{cases} |u_j|=1 & \text{if } j\in\Omega' \\ |u_j|=0 & \text{if } j\notin\Omega'. \end{cases} \tag{3.5}$$

The attraction basin of $B$ is

$$\mathscr{B}(\Omega')=\{|u_j|^2>\xi \text{ if } j\in\Omega' \qquad 0<|u_j|^2<\xi \text{ if } j\notin\Omega'\}. \tag{3.6}$$

Thus, we have $2^N$ local attractors of the unperturbed system. For some $\varepsilon$, in some $\delta$-neighbourhood $V_\delta(B(\Omega'))$

$$V_\delta=\{u\colon \mathrm{dist}(u, B(\Omega'))<\delta\} \tag{3.7}$$

where

$$\mathrm{dist}(u,B)=\inf_{v\in B}\sup_j|u_j-v_j| \tag{3.8}$$

one can develop a perturbation theory using standard results on invariant manifolds. Let some set $\Omega'$ be fixed. In the described neighbourhood (for enough small $\delta$, however, positive $\delta$ does not depend on $\varepsilon$ as $\varepsilon\to0$) one uses special coordinates $(\phi, \boldsymbol{v})$, where

$$u_j=\exp(\mathrm{i}\phi_j)(1+v_j) \qquad j\in\Omega' \tag{3.9}$$

$$u_j=v_j \qquad j\notin\Omega'. \tag{3.10}$$

Here the vector $\phi$ has $N'=|\Omega|$ components.

For $j \in \Omega'$ one sets, in addition, in order to obtain a one-to-one map $\boldsymbol{u} \to (\phi, \boldsymbol{v})$, that

$$\operatorname{Im} v_j = 0 \qquad j \in \Omega'. \tag{3.11}$$

This changing of coordinates has a simple geometric meaning; namely, if (3.11) holds, then the point $\tilde{\boldsymbol{u}}(\phi)$

$$\tilde{\boldsymbol{u}}(\phi) = \begin{cases} \exp(i\phi(j)) & j \in \Omega' \\ 0 & j \in \Omega' \end{cases} \tag{3.12}$$

is the nearest point of $B$ with respect to $u_j$, and $\max|v_j|$ is the distance between $B(\Omega')$ and $u$. It is easy to check that if $\delta$ small, then $\boldsymbol{u} \to (\phi(\boldsymbol{u}), \boldsymbol{v}(\boldsymbol{u}))$ is a one-to-one map with an inverse map $\phi, \boldsymbol{v} \to \boldsymbol{u}(\phi, \boldsymbol{v})$. Using these coordinates in $V_\delta$ one reduces (2.1) to the form (following a standard procedure [5, Ch. 6, 9])

$$\boldsymbol{v}_t = A\boldsymbol{v} + \boldsymbol{F}(\phi, \boldsymbol{v}) \tag{3.14}$$

$$\phi_t = \Phi(\phi, \boldsymbol{v}). \tag{3.15}$$

The linear operator $A$ has the form

$$(A\boldsymbol{v})_j = 2f'(1)v_j \qquad j \in \Omega' \tag{3.16}$$

$$(A\boldsymbol{v})_j = f(0)v_j \qquad j \notin \Omega' \tag{3.17}$$

where the functions $\Phi$ and $F$ can be written in the form

$$\Phi_j = \lambda_j \operatorname{Im} g_j(\boldsymbol{v}) \qquad j \in \Omega' \tag{3.18}$$

$$F_j(\phi, \boldsymbol{v}) = f((1+v_j)^2)(1+v_j) - 2f'(1)v_j + \lambda_j \times (1+v_j) \times \operatorname{Re} g_j(\boldsymbol{u}(\phi, \boldsymbol{v}))$$

$$= h(v_j) + \lambda_j(1+v_j) \operatorname{Re} g(\boldsymbol{u}_j(\phi, \boldsymbol{v})) \qquad \text{for } j \in \Omega' \tag{3.19}$$

$$F_j(\phi, \boldsymbol{v}) = [f(|v_j|^2) - f(0) + \lambda_j g_j(\boldsymbol{u}(\phi, \boldsymbol{v}))]v_j \qquad \text{for } j \notin \Omega' \tag{3.20}$$

and where $g$ is defined by

$$g_j = \sum_k [iJ^1_{jk} u_k(t) + J^2_{jk} q(u_k(t))]. \tag{3.21}$$

*Lemma 3.1.* Suppose that the following inequality holds at the initial moment

$$|\boldsymbol{v}(0)| = \max_j |v(0)| < \delta. \tag{3.22}$$

If $\delta$ is small enough, there exists a number $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$ there is an *a priori* estimate

$$\max_j |v_j(t)| < \min[\delta, C\delta \exp(-\varkappa t) + \varepsilon]. \tag{3.23}$$

*Proof.* See appendix.

Now one can easily prove the existence of an invariant manifold $M(\Omega')$ by standard methods [5, Ch. 6]. In fact, here the operator $A$ has the property of trivial exponential dichotomy

$$|(\exp At)\boldsymbol{v}| < c \exp(-\varkappa t)|\boldsymbol{v}|. \tag{3.24}$$

Thus, here we deal with system (3.14) and (3.15) having a typical form with the slow variable $\phi$ and the quick one $\boldsymbol{v}$. Following [5, 6] one can check (using lemma 3.1) that this manifold $M(\Omega')$ has the equation

$$\boldsymbol{v}=\sigma(\phi) \tag{3.25}$$

and the following estimates hold:

$$|\sigma|<c\varepsilon \qquad |\sigma'_\phi(\phi)|<c\varepsilon. \tag{3.26}$$

This manifold is stable and attracts all solutions from $V_\delta$ with an exponential rate. Thus all local attractors and invariant sets in $V_\delta$ lie on invariant manifold $M(\Omega')$. The dynamics of $\phi$ on $M$ are defined by the ordinary equations

$$\mathrm{d}\phi_j(t)/\mathrm{d}t=\Phi_j(\phi, \sigma(\phi)). \tag{3.27}$$

Using estimate (3.27), one can obtain from lemma 3.1 the following auxiliary assertion.

*Lemma 3.2.* The field $\Phi$ has at least $C^1$-smoothness and has the form

$$\Phi=\boldsymbol{Z}^0(\phi,\Omega')+\boldsymbol{Z}^1(\phi,\Omega') \tag{3.28}$$

where the correction $\boldsymbol{Z}^1$ satisfies the estimate

$$|\boldsymbol{Z}^1|_1=|\boldsymbol{Z}^1|_{C^1}\leqslant c\varepsilon. \tag{3.29}$$

The field $\boldsymbol{Z}^0$ has the explicit form

$$Z_j^0=\lambda_j \operatorname{Im} g_j(\tilde{\boldsymbol{u}}(\phi), \mathbf{0}) \tag{3.30}$$

where $\tilde{\boldsymbol{u}}$ is defined by equation (3.12).

*Proof.* It is obvious from the following simple deductions. The function $\Phi$ falls into the two contributions

$$\Phi=\boldsymbol{Z}^0+\boldsymbol{Z}^1=\lambda_j \operatorname{Im} g(\tilde{\boldsymbol{u}}(\phi),\mathbf{0})+\lambda_j[(\operatorname{Im} g(\tilde{\boldsymbol{u}}(\phi),\boldsymbol{v})-\operatorname{Im} g(\tilde{\boldsymbol{u}}(\phi),\mathbf{0})]. \tag{3.31}$$

It is clear from (3.31) that

$$\begin{aligned} &|\boldsymbol{Z}^1|_0\leqslant \varepsilon C|\sigma|\leqslant c\varepsilon^2 \\ &|\boldsymbol{Z}^1|_1\leqslant \varepsilon C(|\sigma|+|\sigma'|)\leqslant c\varepsilon^2. \end{aligned} \tag{3.32}$$

To conclude this section, one notices, that the field $\boldsymbol{Z}^0$ can be rewritten in the more explicit form

$$Z_j^0(\phi)=\varepsilon_j\left\{\operatorname{Re}\left[\sum_k J^1_{jk}\exp(\mathrm{i}\phi_k)\right]+\sum_k J^2_{jk}\phi_k\right\}\left\{1+\sum_k\left[|J^1_{jk}|+|J^2_{jk}|\right]\right\}^{-1}. \tag{3.33}$$

This will be used in section 4.

## 4. Solution of inverse problem

In this section we solve the following inverse problem. Let some local attractor $\mathscr{U}$ (which attracts each point from a neighbourhood $V(\mathscr{U})$) be given. Let this set $\mathscr{U}$ be structurally stable (for example, a hyperbolic set [19]) and be defined by the flow

$$\mathrm{d}y/\mathrm{d}t = Y(y) \qquad y = (y_1, y_2, \ldots, y_n). \tag{4.1}$$

Our aim is to find coefficients $J^s_{jk}$ such that system (2.1) has the local attractor $B(\Omega', J)$ which is close to $\mathscr{U}$. To be more precise, $B(\Omega', J)$ and $\mathscr{U}$ should be connected with a homeomorphism $T$ which maps trajectories $y(t) \in \mathscr{U}$ of (4.1) into trajectories $B$, and $T$ should be close to the identity $I: \mathscr{U} \to \mathscr{U}$

$$|T - I| < \delta. \tag{4.2}$$

To solve this inverse problem one transforms (4.1) into the system

$$\mathrm{d}y/\mathrm{d}t = \bar{Y}(y, z) \tag{4.3}$$

$$\mathrm{d}z/\mathrm{d}t = Z(y, z) \tag{4.4}$$

with a close attractor. This new system has a higher dimension and also has an invariant manifold $\tilde{M}$. On this manifold the flow defined by (4.3) and (4.4) can be reduced to the system

$$\mathrm{d}y/\mathrm{d}t = \tilde{Y}(y) \tag{4.5}$$

where

$$|\tilde{Y} - Y| < \delta_1 \tag{4.6}$$

in the neighbourhood of $V$. Then, using the persistence hyperbolic sets theorem [19, 20], one obtains that system (4.5) has an attractor $\tilde{\mathscr{U}}$ such that

$$T(\mathscr{U}) = \tilde{\mathscr{U}} \tag{4.7}$$

and (4.2) holds.

To construct this transformation, let us consider (4.1) in a box $\Pi$ which contains $\mathscr{U}$. One can set

$$\Pi = \{0 < y_i < 2\pi\}. \tag{4.8}$$

Outside $\Pi$ one can continue $Y$ so that $Y$ is periodic. Then

$$Y(y) = \sum_{k \in \mathbb{Z}^n} Y(k) \exp(\mathrm{i}k \cdot y) \tag{4.9}$$

where

$$k = (k_1, k_2 \ldots k_n) \qquad k_s \in \mathbb{Z} \qquad k \cdot y = \sum k_s y_s. \tag{4.10}$$

One can cut off row (4.9) so that

$$\tilde{Y}(y) = \sum_{|k| < K} Y(k) \exp(\mathrm{i}k \cdot y). \tag{4.11}$$

The difference between the fields $\tilde{Y}$ and $Y$ will be small if $K$ is large enough. One considers the system

$$\mathrm{d}y_j/\mathrm{d}t = \mathrm{Re} \sum_{|l| < K} Y_j(l) \exp(\mathrm{i}z(l)) \tag{4.12}$$

$$\mathrm{d}z(k)/\mathrm{d}t = \mathrm{Re} \sum_{|l| < K} (k \cdot Y(l)) \exp(\mathrm{i}z(l)) - \alpha\left(z(k) - \sum k_s y_s\right) \tag{4.13}$$

where $l$ is the multi-index $(l_1, \ldots, l_n)$, $\alpha > 0$.

System (4.12) and (4.13) has the invariant manifold $M$

$$z(l)=\sum_s l_s y_s. \tag{4.14}$$

Flow (4.12) and (4.13) on $M$ can be reduced to the form (4.5). Setting $w(l)=z(l)-l\cdot y$, one finds

$$\mathrm{d}w(l,t)/\mathrm{d}t=-\alpha w(l,t). \tag{4.15}$$

Thus

$$|w(l,t)|<c\exp(-\alpha t) \tag{4.16}$$

and one concludes that $M$ is a stable manifold and trajectories (4.12) and (4.13) tend to those of (4.5) with an exponential rate.

Finally, here one obtains the following auxiliary assertion.

*Lemma 4.1.* If system (4.5) has a hyperbolic (more generally, persistent under $C^1$-perturbations) attractor $\mathcal{U}$, then system (4.12) and (4.13) also has a hyperbolic attractor $\mathcal{U}$ with close trajectories. Moreover, relation (4.7) holds together with estimate (4.2).

To solve the inverse problem, let us now choose the coefficients $J^s_{jk}$ in a special way.

Let us mark out the nodes $j_1,\ldots,j_n$ from $\Omega$ setting $\Omega_1=\{j_1,\ldots,j_n\}$. Let $s(j)$ be the index of node $j$ from $\Omega_1$, for instance if $j=j_1$, then $s=1$, *if* $j=j_2$, then $s=2$, etc. Furthermore, let $j(\boldsymbol{k})$ be a one-to-one mapping which maps each multi-index $\boldsymbol{k}=(k_1,k_2,\ldots,k_n)$, $|\boldsymbol{k}|<K$ into the node $j\in\Omega_2$, where the set $\Omega_2\subset\Omega\backslash\Omega_1$. The corresponding inverse map will be denoted by $\boldsymbol{k}(j)$. Let $N_2=|\Omega_2|$.

Let us now set

$$\tilde{J}^1_{jj'}=Y_j(\boldsymbol{k}(j')) \qquad j\in\Omega_1 \qquad j'\in\Omega_2 \tag{4.17}$$

$$\tilde{J}^1_{jj'}=\sum_m k_m(j)\,Y_m(\boldsymbol{k}(j')) \qquad j,j'\in\Omega_2 \tag{4.18}$$

and in the other cases $\tilde{J}^1_{jj'}\equiv 0$. For $\tilde{J}^2_{jj'}$ one uses the forms

$$\tilde{J}^2_{jj'}=-\alpha\delta_{jj'} \qquad j,j'\in\Omega_2 \tag{4.19}$$

$$\tilde{J}^2_{jj'}=\alpha k_{s(j')}(j) \qquad j\in\Omega_2 \qquad j'\in\Omega_1 \tag{4.20}$$

and $\tilde{J}^2_{jj'}$ is zero in the other cases.

After this, one sets

$$\beta_j\tilde{J}^k_{jj'}-\delta_1<J^k_{jj'}<\beta_j\tilde{J}^k_{jj'}+\delta_1 \tag{4.21}$$

where

$$\beta_j=\left(\varepsilon_j-\sum_{j'}[|\tilde{J}^1_{jj'}|+|\tilde{J}^2_{jj'}|]\right)^{-1} \tag{4.22}$$

and $\delta_1$ is a small positive number.

One also sets $L=|\Omega'|$, $\Omega'=\Omega_1\cup\Omega_2$. If we identify the variables $\phi$ and the variables $y,z$ by

$$\{\phi_1,\ldots,\phi_L\}\Longleftrightarrow\{y_1,\ldots,y_n,\{z(\boldsymbol{k})\},\boldsymbol{k}=\boldsymbol{k}(j),j\in\Omega_2\}$$

then system

$$d\phi/dt = Z^0(\phi) \tag{4.23}$$

coincides with (4.12) and (4.13) with sufficient accuracy, if all $\varepsilon_j$ are sufficiently small.

Finally, the principal result of sections 2–4 is the following.

*Theorem 4.1.* Let

$$dy/dt = Y^{(k)}(y) \qquad k=1,2,\ldots,P \tag{4.24}$$

be some prescribed systems of ODE, where $y$ lies in the cub $\{0<y_i<2\pi\}$, with some prescribed hyperbolic attractors $\mathcal{U}_k$. Let the parameter $\varepsilon$ be small enough and the number $N$ be large enough such that $0<\varepsilon_j<\varepsilon$ for all $j$.

Then there exist coefficients $J^k_{jj'}$ such that system (2.1) has equivalent attractors $\tilde{\mathcal{U}}_k$, $\tilde{\mathcal{U}}_k \propto \mathcal{U}_k$ (in the sense that there exist homeomorphisms $T$ described above satisfying (4.7) and (4.2)).

The attraction basins of $\tilde{\mathcal{U}}_k$ contain the domains

$$B(\Omega_k) = \{|u_j|^2 > \xi + \delta_2, \text{ if } j \in \Omega_k, |u_j|^2 < \xi - \delta_2, \text{ if } j \notin \Omega_k\} \tag{4.25}$$

where $\delta_2$ is sufficiently small and $\Omega \subset \Omega$ are subdomains of $\Omega$.

*Proof.* See appendix.

## 5. Some remarks and discussion

Let us discuss briefly the case of random connections $J$. Assume the coefficients $J^p_{kj}$ have independent Gaussian distributions. Then the following result is trivial.

*Theorem 5.1.* Let the number $P$ of the prescribed hyperbolic local attractors $\mathcal{U}_P$ be fixed. Let $\rho_N$ be the probability of obtaining, with help of (2.1), the attractors $\tilde{\mathcal{U}}_P$ which are equivalent to $\mathcal{U}_P$, $\mathcal{U}_P \propto \tilde{\mathcal{U}}_P$ (this means that trajectories are connected with the help of some homeomorphisms). Then the probability $\rho_N$ tends to 1 as $N$ tends to infinity:

$$\rho_N \to 1 (N \to \infty). \tag{5.1}$$

Let us turn now to the problem of so-called parasite attractors. Suppose the prescribed systems (4.1) are structurally stable, for example, they satisfy the Smale $A$-axiom [19, 20]. Notice, except for the prescribed attractors, dynamical system (2.1) can have a number of other attractors. These attractors can be called parasite attractors.

It is easy to suggest a method which allows one to take off these parasite attractors. We can construct systems which have prescribed (and only prescribed!) local attractors coinciding with local ones for prescribed systems (4.1). One changes the first contribution in the right-hand side of equation (2.1). Consider

$$du_j(t)/dt = -\partial/\partial u_i^* L[uu^*] + \lambda_j \left\{ \sum_k [iJ^1_{jk} u_k(t) + J^2_{jk} q(u_k(t))] \right\} u_j(t) \qquad u_j(t) \in \mathbb{C}. \tag{5.2}$$

Here $J$, $\lambda$ should satisfy the same conditions as in section 2, and $L$ is some smooth real function of the vector $\boldsymbol{uu}^*$ with components $|u_i|^2$. Suppose $L$ is bounded below

$$L[\boldsymbol{uu}^*] \geqslant -c \tag{5.3}$$

and $L$ has exactly $P$ prescribed local non-degenerate minima in $P$ different configurations

$$|u_i|^2 = \xi_i^p \qquad p=1\ldots P \qquad i=1,\ldots,N \tag{5.4}$$

where $\xi_i^p$ is zero or unity: $\xi_i^p=0$ or 1. Assume also that

$$\xi_i^k=1 \quad \text{for some } k \qquad \text{implies} \qquad \xi_i^p=0 \quad \text{for all } p\neq k. \tag{5.5}$$

Such functions $L$ can be constructed very easily. One can set

$$L=-H(x_1, x_2 \ldots x_N) \qquad x_i=u_i u_i^* \tag{5.6}$$

where $H$ is some unknown potential relief.

Let us connect points $\xi^P$ with the help of some curve $Z$ which has no self-intersections. Moreover, let us suppose that the relief $H$ has a narrow and sharp ridge along this curve. In other words, all the maxima of $H$ are concentrated in some small neighbourhood of $Z$ (if $H$ is a mountain height then all highest mountains lie at $Z$).

Clearly, along $Z$ the function $H$ can be considered as a function $s$ where $s$ is the arclength of $Z$. Now it is easy to construct a function $H(s)$ with $P$ prescribed maxima and having only non-degenerate extrema.

To investigate the behaviour of solutions (5.2), one can use the same arguments as at the end of the introduction and in the proof of theorem 4.1 (see appendix). Finally, it is clear that, for some time $t$ (depending on initial data), the 'generic' solution reaches some neighbourhood of the prescribed configurations. All other configurations are excluded. More exactly, we know from [5] that non-degenerate saddle points and maxima of the potential energy $L$ attract only trajectories lying on some embedded submanifolds. In fact, this is a consequence of theorem 6.1.9 and 6.1.10 from [5] since these points are unstable hyperbolic equilibria of dynamic flow (5.2) with $\lambda=0$. Thus, the measure of the union of these bad (non-generic) trajectories is 0. If a generic trajectory exists in some small neighbourhood of a prescribed configuration, then the following analysis of the trajectory behaviour does not depend on the detail of function $L$ form. Thus, we can repeat it, following sections 3 and 4.

Finally, if coefficients $J$ are chosen as in section 4, we obtain only prescribed attractors. These attractors were denoted above $B(\Omega_P)$ where $\Omega_P$ is the set of nodes $i$ such that $\xi_i^P=1$. Hypothesis (5.5) yields that attractors $B(\Omega_P)$ can be constructed independently.

These results hold only if systems (4.1) (which should be simulated by our coupled oscillators system) are structurally stable. Such situation seldom occurs in real applications. For example, even a Lorenz system is not hyperbolic and is not structurally stable. Stability properties of this system were studied in [21–23]. Nonetheless, even in the general case, when the prescribed attractors are not hyperbolic, one can obtain some results. These depend on the smoothness of the right parts $Y(y)$ of (4.1). If $Y\in C^\infty$ then one can guarantee that the trajectories of (2.1) lie near some prescribed attractor $\mathcal{U}$ for time $\propto O(\lambda^{-K})$ where $K>0$ is an arbitrary integer. However, we cannot assert (at least, from a rigorous mathematical point of view) that these trajectories remain near $\mathscr{U}$ forever. The proof uses simple asymptotical calculations together with some *a priori* estimates (these can be obtained as in [13]).

Our solution of the inverse problem allows us to obtain unknown coefficients $J$ giving prescribed attractors by explicit and closed forms (4.17)–(4.21). However, there is another approach which allows one to calculate $J$ by computer and, thus, to check this model. This approach helps to determine the coefficients $J$ for the important (from a practical point of view) situation when the explicit form of the right-hand sides of (4.1) are unknown, but we do have calculated or experimental trajectories

$$y(t_0), y(t_0+h), \ldots, y(t_0+(n-1)h), \qquad y(t_0+nh) \qquad y\in\mathbb{R}^m \tag{5.7}$$

of the prescribed system (4.1). Suppose we know in advance that $y \in C^m$ with $s>1$, moreover, without loss of generality, let us assume $y_i(t)\in[0, 2\pi]$. Our approximate method consists of the following steps:

(1) Choose some subset $\Omega'$ of the lattice $\Omega$ containing a sufficient number $N'=|\Omega'|$ of nodes $j$. This number should be chosen so that the attractor $B(\Omega')$ (which was described above, in section 4) can coincide with the attractor of (4.1). Clearly, $N'\propto a(s)^m$ where $a(s)$ depends on the smooth parameter $s$ (which defines the cut-off constant $K$, see (4.11)) and $a(s)$ decreases as a function of $s$. Finally, let us choose some nodes $i_1, i_2, \ldots, i_m$ which will contain angular variables $\phi_i$. The corresponding values of $\phi$ will be used below for a comparison with data (5.7).

(2) Consider the comparison functional

$$S=(nm)^{-1}\sum_k^n\sum_i^m (y_i(t_0+kh)-\phi_i(t_0+kh))^2. \tag{5.8}$$

Clearly that the function depends on coefficient $J$. If theorem 4.1 holds then a minimum of $S(J)$ is zero (practically, it should be a sufficiently small number). Thus, unknown coefficients $J$ can be found as a minimizer of $S$.

In sections 2–4, noise is not taken into consideration. Clearly, from [24], the multiplicative noise influence can be investigated. Consider the Langevin equation

$$\mathrm{d}u_j(t)=f(|u_j(t)|^2)u_j(t)+\lambda g_j\cdot u_j(t)+\delta\sum_{k=1}^M H_i^{(k)}(u)\,\mathrm{d}\omega_k \tag{5.9}$$

where $\omega_k$ are standard Wiener processes and $H^{(k)}$ are some vector fields, $H\in\mathbb{C}$. The number $\delta$ is a positive small parameter. Under the influence of noise a trajectory $\boldsymbol{u}(t)$ (lying at first in the attractor $B(\Omega')$) can leave a neighbourhood $V_B$ of $B$. In particular, the trajectory can enter some set $Q$. This set consists of states $\boldsymbol{u}$ which correspond to some non-appropriate behaviour: for example, it can correspond to a fatal error in the system behaviour. One can estimate the corresponding probability $\rho(B, Q, \delta)$ (otherwise, estimate stochastic stability) as follows.

Our analysis holds also for general dynamical systems

$$\mathrm{d}\boldsymbol{u}=\boldsymbol{F}(\boldsymbol{u})\,\mathrm{d}t+\sum_{k=1}^M \boldsymbol{H}^{(k)}(\boldsymbol{u})\,\mathrm{d}\omega_k \tag{5.10}$$

where $\boldsymbol{F}, \boldsymbol{H}^{(k)}\in C^\infty$ are smooth fields.

Let us consider the probability $\rho(B, Q, \delta)$. This is the probability of travelling along some trajectory from the attractor $B$ to the 'bad' set $Q$, for a time interval $[O, T]$. The theory of small random perturbations of dynamical systems gives [24]

$$R(B, Q) = \lim_{\delta \to 0} \delta^2 \ln \rho(B, Q, \delta)$$
$$= \inf_T \inf \int_0^T \langle D^{-1}(\mathrm{d}u/\mathrm{d}t - F(u)), \mathrm{d}u/\mathrm{d}t - F(u) \rangle \, \mathrm{d}t. \tag{5.11}$$

The matrix $D$ in (5.8) is the matrix from the corresponding Fokker–Planck equation

$$D_{ij}(u) = \sum_{k=1}^{M} H_i^{(k)}(u) H_j^{(k)}(u) \tag{5.12}$$

inf in (5.8) is taken over all trajectories from $B$ to $Q$. Notice that $R$ from (5.8) is positive if and only if there exists a trajectory from $B$ to $Q$ defined by the equations

$$\mathrm{d}u/\mathrm{d}t = F(u) + \sum_k v_k(t) H^{(k)} \tag{5.13}$$

for some functions $v_k$.

The basic C Lobry result [25] shows (we omit some details of interest only to mathematicians) that, generically, $R$ is positive, thus

$$\rho(B, Q, \delta) \propto \exp(-c(N, B, Q)\delta^{-2}) \text{ as } \delta \to 0. \tag{5.14}$$

The physical meaning of this result is very simple. It means that (in the generic case) an arbitrary system simulating the prescribed behaviour gives non-zero probability of a 'fatal error', if the size $N$ of the system is bounded. Only in the thermodynamic limit (large systems) can one create a system with stable behaviour. Of course, from the physical point of view, this is a trivial assertion.

## 6. Summary

The investigation of attractors is, probably, the central problem of nonlinear dissipative system theory. However, it is a formidable task! In fact, most rigorous papers estimate only the attractor dimension. In reality, information has been obtained only for systems with Lyapunov functions or monotone systems. Generically, the large time behaviour of such systems is sufficiently trivial: it is a limit cycle or a rest point.

Here a simple class of systems is described where attractors can be complicated. The key idea of the proposed construction is to take highly symmetrical unperturbed systems with Lyapunov functions and a small perturbation which breaks this symmetry.

This paper develops the author's previous works where reaction–diffusion systems with complicated behaviour were described by rigorous methods [12–14]. The model investigated here reminds one of coupled oscillator systems. This model, due to non-local interactions, has a richer set of attractors, than in [13, 14], where the attractor dimension was bounded from above.

Finally, roughly speaking, it is shown that some artificial models can have arbitrary prescribed local attractors (if we neglect mathematical details connected with the hyperbolicity). For such models, we can solve the inverse problem: namely, by changing model parameters, we can, with the help of explicit forms such as (4.17)–(4.22), obtain arbitrary prescribed attractors.

These models can be constructed with the help of small non-local perturbations of the systems, which are invariant under some sufficiently rich symmetry group.

The suggested method can probably be applied to other problems, for example the perturbed Ginzburg–Landau equation or the Swift–Hoenberg equation.

**Acknowledgments**

This paper was stimulated by problems connected with animal behaviour. The author is very grateful to his daughter M Vakulenko for giving the possibility to investigate the behaviour of frogs, turtles, tritons, cats and others. The author also thanks professor J-L Lions for the interest shown in these studies on prescribed attractors, which stimulated this paper, and also for his interpretation of the results [23] that helped simplify the statement.

## Appendix

This appendix contains the proofs of various assertions made in the main text.

*Proof of proposition 2.1.* We can use the maximum principle following [26]. Let us write the equation for amplitudes $r_j = |u_j|$, which takes the form

$$\mathrm{d}r_j/\mathrm{d}t = f(r_j)r_j + \lambda h_j = F_j(r) \tag{A.1}$$

where $h$ is a correction which is bounded.

Let us prove that $U = \{r\colon r_j < C\}$ is an invariant region. As follows from [26], it is necessary to prove that

$$\boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{n} = \sum F_j n_j < 0 \qquad \text{if } \boldsymbol{r} \in \partial U \tag{A.2}$$

where the vector $\boldsymbol{F}$ defines the right-hand side of (A.1) and $\boldsymbol{n}$ is the normal vector of the boundary $\partial U$. Consider the face of the cube $U$ such that $\boldsymbol{n} = (1, 0, 0, \ldots, 0)$. Then $r_1 = C$, $r_j \leqslant C$ and $\boldsymbol{F} \cdot \boldsymbol{n}$ is simply $F_1$. So, we should estimate $F_1$ under the conditions

$$r_1 = C \qquad \text{and} \qquad r_j \leqslant C. \tag{A.3}$$

One notices that in $U$

$$|h_j| < C|g_j| \tag{A.4}$$

where $g_j$ is defined by (3.21). One has from (2.4), (2.5) and (2.7) that

$$|g_j| = \left|\sum_k \{J^1_{jk}u_k + J^2_{jk}q(u_k)\}\right| < c_1 C\left[\sum_k |J^1_{jk}| + |J^2_{jk}|\right]. \tag{A.5}$$

Thus

$$|\lambda_j \cdot h_j| < \varepsilon c_2 C^2. \tag{A.6}$$

On the other hand, one finds (using (2.3)) that when (A.3) holds, one has (as follows from the assumptions on $f$, see section 2)

$$f(r_1)r_1 = f(C^2)C < -\delta. \tag{A.7}$$

Thus, combining (A.7) and (A.6), one obtains

$$F_1 < \varepsilon c_2 C^2 - \delta < 0 \tag{A.8}$$

for small $\varepsilon$. This completes the proof.

*Proof of lemma 3.1.* Using proposition 2.1 and estimates (2.3) and (2.12) one finds

$$\mathrm{d}v_j(t)/\mathrm{d}t \leqslant 2f'(1)v_j(t) + c_1 v_j^2(t) + c_2\varepsilon \qquad j\in\Omega' \tag{A.9}$$

and for $j\notin\Omega'$

$$\mathrm{d}v_j(t)/\mathrm{d}t = f(0)v_j(t) + \tilde{h}(v_j(t)) \qquad |\tilde{h}(v)| \leqslant c(|v|^2+\varepsilon) \tag{A.10}$$

In the first inequality $v$ is real thus the right-hand side of (A.9) is negative for small $\varepsilon$, and $v_j=\delta$. Thus, the set $\{v_j\leqslant\delta, j\notin\Omega'\}$ is the invariant region.

The second equation entails (one sets $\varkappa = 1/2\min(-2f'(1), -f(0))$)

$$\mathrm{d}/\mathrm{d}t\,|v|^2 \leqslant -2\varkappa|v|^2 + c^2(4\varkappa)^{-1}\varepsilon^2 - \varkappa|v|^2 + c|v|^3 = p$$

and $p<0$ on $\partial V_\delta$. Thus, $V_\delta$ is the invariant region. Therefore, one can conclude that

$$|v_j(t)| < \delta \text{ for any } t \text{ and } j \tag{A.11}$$

if this holds at the initial moment. Then, using estimate (A.11), one finds from (A.9) and (A.10), that

$$\mathrm{d}/\mathrm{d}t\,|v_j(t)|^2 \leqslant -\varkappa|v_j(t)|^2 + c\varepsilon^2. \tag{A.12}$$

The last relation implies (3.23), at once completing the proof.

*Proof of Theorem 4.1.* Excepting for the assertion regarding the attraction basins, all other assertions are already proved in section 4. Thus, one can suppose that there exist a set $\Omega_k$ and connections $J$ such that the corresponding attractors $B(\Omega_k, J)$ coincide with $\tilde{\mathscr{U}}_k$ and $\tilde{\mathscr{U}}_k \propto \mathscr{U}_k$. Also one can assume that the attraction basins $\tilde{\mathscr{U}}_k$ contain the small neighbourhood $V_\delta$ of $\tilde{\mathscr{U}}_k$. Let us prove that if initial data lie in $B(\Omega_k)$ then, beginning at some moment $t_0$, the solution $\boldsymbol{u}(t)$ is given at the small neighbourhood $V_\delta$ of the set $Q=\{|u_j|=1 \text{ for } j\in\Omega_k, |u_j|=0 \text{ for } j\notin\Omega_k\}$.

Let us compare the trajectory $\boldsymbol{u}(t,\varepsilon)$ of (2.1) with the unperturbed trajectory $\boldsymbol{u}(t,0)$ supposing that at the initial moment the corresponding initial data coincide.

It is easy to check using proposition 2.1 that

$$|\boldsymbol{u}(t,\varepsilon) - \boldsymbol{u}(t,0)| < c\varepsilon\exp(ct). \tag{A.13}$$

The unperturbed trajectory reaches $V_\delta$ during the time interval $[0, t_1]$, where $t_0$ does not depend on $\varepsilon$. Thus, estimate (A.13) yields the same fact for $V_\delta$ and $\boldsymbol{u}(t,\varepsilon)$, if $\varepsilon$ is small.

Now the perturbation theory (described in sections 3 and 4) is used, completing the proof.

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